Better synchronizability predicted by a new coupling method

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In this paper, inspired by the idea that different nodes should play different roles in network synchronization, we bring forward a coupling method where the coupling strength of each node depends on its neighbors' degrees. Compared with the uniform coupled method and the recently proposed Motter-Zhou-Kurths method, the synchronizability of scale-free networks can be remarkably enhanced by using the present coupled method, and the highest network synchronizability is achieved at $\beta = 1$ which is similar to a method introduced in [AIP Conf. Proc. 776, 201 (2005)].

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I. INTRODUCTION

Many collective dynamics in social, biological and communication systems can be properly described by complex networks. These networks exhibit complex topological properties such as the small-world effects and the scale-free properties [1, 2, 3, 4]. Many kind of network models have been made to embody these properties. The so-called small-world networks are the intermediates of regular lattices and random networks in structure but bear both characters of the two kind of networks, that is, they have small average distance as random networks and large clustering coefficient as regular ones [5]. The scale-free networks are a kind of small-world networks with degree distribution obeying a power-law form. A scale-free network can be created by successively adding new nodes to the network and connecting them with the already existing ones by the preferential attachment rule [6].

The interesting topological properties of complex networks make the dynamics taking place on them much different from those on regular or random ones. For example, coupled dynamical oscillators on small-world networks are much easier to synchronize than on regular lattices, and increasing the proportion of shortcuts of networks will make the oscillators more synchronizable [7, 8, 9, 10]. It has also been observed that the more heterogeneous of the network degree distribution is the harder for the oscillators on the network to synchronize [11]. Therefore, generally speaking, networks with short average distance and homogeneous degree distribution will make the oscillators on them more synchronizable [11, 12, 13, 14].

Very recently, motivated by practical requirements and theoretical interest, numbers of researches have begun to study how to enhance the network synchronizability, especially for scale-free networks [15, 16, 17, 18, 19, 20]. The method proposed by Zhao *et al.* [18] can sharply reduce the maximal betweenness thus enhance the network synchronizability, but it will bring some economic and technologic problems since the network structure is slightly changed. The method proposed by Chavez *et al.* keeps the network topology unchanged, while adding some weight into the system [19, 20]. However, to compute the weight, this method needs the global structural information, which is usually unavailable in huge communication systems. Therefore, in this paper, we keep the network topology unchanged, and concentrate on the coupling method using only the local information. The Motter-Zhou-Kurths (MZK) method [15] is a typical example, in which the coupling strength from a node i is inverse to its degree k_i. In MZK method, every neighbor of a node has the same influence (coupling strength) to this node. However, in real networks, different nodes may have different influences. For example, in society, some people have strong influence on others in some aspect but they are not influenced at the same level. Another impressing phenomenon is that in the World Trade Web, the small countries' economies fluctuate with the powerful countries tightly, but the contrary does not occur [21]. Thus here, based on the assumption that different nodes play different roles, we adjust the influencing strength of each node receiving from their neighbors according to the neighbors' degrees. That is, a node is not influenced by its neighbors equally. It is found that oscillators on scale-free networks coupled in this way can have much stronger propensity for synchronization than in the previous ways, with the exception of a recently introduced method [16, 22] , where the former is similar to $beta = 1$ in our model and is shown here to be the most efficient method based on the information provided by the degree of nearest neighbors only.

This paper is organized as follow: in section 2, the dynamical equations of coupled oscillators and the master stability function will be briefly introduced. In section 3, we will give the simulation and analysis about synchronization of correlated scale-free networks. Finally, we will draw our conclusion in section 4.

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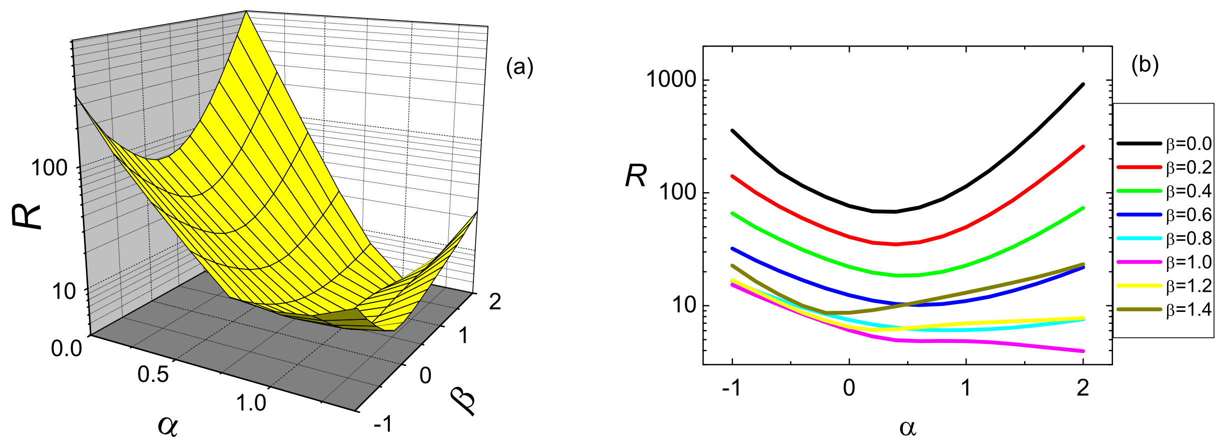

FIG. 1: (color online) (a) R in the parameter plane (α, β). (b) R vs α for different parameter β. The numerical simulations are implemented based on the BA network of size $N = 1024$ and with average degree $\bar{k} = 6$. The data are obtained over 10 independent realizations.

II. THE DYNAMICAL EQUATIONS AND MASTER STABILITY FUNCTION

For a network of N linear coupled identical oscillators, the dynamical equation of each oscillator can be written as

$$\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}^j), \quad i = 1, 2, ..., N, \tag{1}$$

where $\dot{\mathbf{x}}^i = \mathbf{F}(\mathbf{x}^i)$ governs the dynamics of individual oscillator, $\mathbf{H}(\mathbf{x}^j)$ the output function, σ the coupling strength, and G_{ij} the elements of the $N \times N$ coupling matrix. To guarantee the synchronization manifold an invariant manifold, the matrix G should has zero row-sum. Traditionally, the oscillators are coupled symmetrically with uniform coupling strength and the coupling matrix G has the same form as Laplacian matrix L, that is, $G_{ij} = L_{ij}$, where

$$L_{ij} = \begin{cases} k_i & \text{for } i = j \\ -1 & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where k_i is the degree of node i and Λ_i is the set of i's neighbors. Because of the symmetry and the positive semidefinite of L, all its eigenvalues are nonnegative reals and the smallest eigenvalue λ_0 is always zero, for the rows of L have zero sum. And if the network is connected, there is only one zero eigenvalue. Thus, the eigenvalues can be ranked as $\lambda_0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_{N-1}$. According to the criteria of master stability function [23, 24, 25, 26], the network synchronizability can be measured by the eigenratio $R = \lambda_{N-1}/\lambda_1$: The smaller it is the better the network synchronizability and vice versa.

It is later found that networks with high heterogeneity of degree distribution coupled uniformly are hard to synchronize. As mentioned above, to eliminate this problem, Motter, Zhou and Kurths suggested the coupling matrix taking the form $G_{ij} = L_{ij}/k_i^{\beta}$ [15, 16, 17]. This simple change of coupling matrix enhances the network synchronizability sharply, and the optimal condition is $\beta = 1$. And by exploiting the information contained in the load of each edge (i.e. set the off-diagonal elements of the zero row-sum coupling matrix G to be $G_{ij} = l_{ij}^{\alpha}/\sum_{j=1}^{N} l_{ij}^{\alpha}$, where l_{ij} is the load of edge connecting node i and j), further enhancement in synchronization is achieved [19, 20].

Many real-world networks are highly heterogeneous with a few nodes, named hubs, having very large degrees. When using the uniform coupling method, these hubs synchronize first, and slowly the nodes with fewer degree synchronize to them [27]. If the influence of the hubs on the low-degree nodes becomes stronger, the latter will synchronize to the former much easier, obviously, the network synchronizability will be enhanced. Therefore, we argue that not only reducing the communication load of hubs (as did in MZK method), but also increase their influences may further enhance the network synchronization.

Here we take into account the effects of different degrees of nodes on synchronization, that is, a node in complex network is not coupled uniformly by its neighbors but the coupling strength is modulated by k^{α}. Thus the

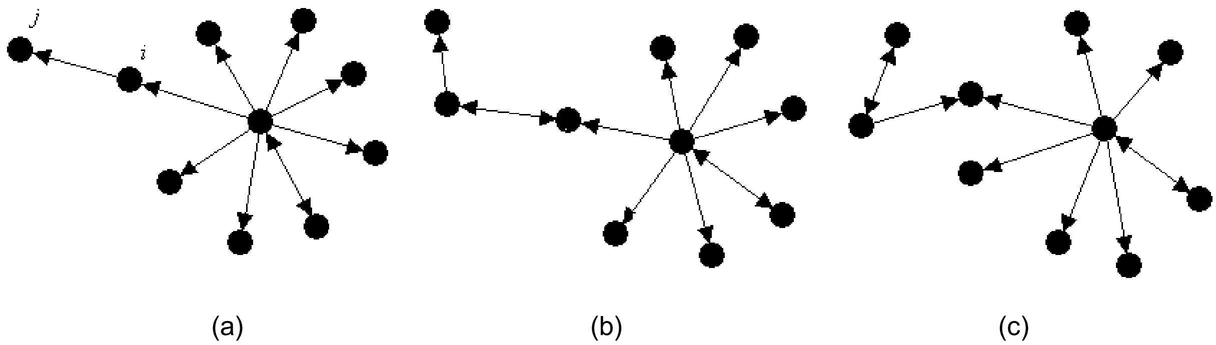

FIG. 2: The sketch maps of three simple equivalent networks, where the arrow form node i to node j indicates the latter receives coupling signal from the former. Their eigenratios are 2 (a), 6.8284 (b) and $+\infty$ (c), respectively.

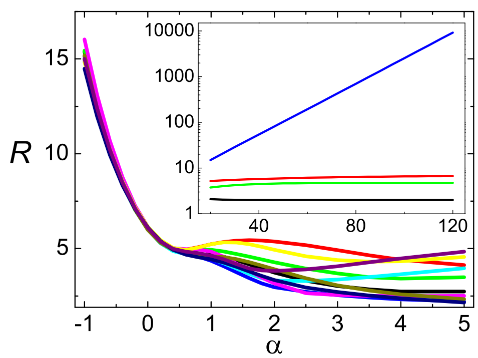

FIG. 3: (color online) The eigenratio R vs parameter α at $\beta = 1.0$ for several BA network configurations of size $N = 1024$ and with average degree $\bar{k} = 6$. Each color represents one configuration.

coupling matrix G takes the form

$$G_{ij} = \begin{cases} S_i/S_i^{\beta} & \text{for } i = j \\ -k_j^{\alpha}/S_i^{\beta} & \text{for } j \in \Lambda_i \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where $S_i = \sum_{j\in\Lambda_i} k_j^{\alpha}$. When $\alpha = \beta = 0$, this coupling scheme degenerates to the uniform coupling scheme [23], the case of $\alpha = 0$ corresponds to the MZK method [15], and the case of $\beta = 1$ is equivalent to the one introduced in the ref. [16] (see the Eq.(15) for details).

Using a similar method to the one proposed by Motter *et al.* [17], we next prove that all the eigenvalues of matrix G are real. Note that, Eq. (3) can be written as

$$G = DL', \tag{4}$$

where $D = \mathrm{diag}\{k_1^{-\alpha}S_1^{-\beta}, k_2^{-\alpha}S_2^{-\beta}, ..., k_N^{-\alpha}S_N^{-\beta}\}$ is a diagonal matrix, and $L' = (L'_{ij})$ is a symmetric zero row-sum matrix, whose off-diagonal elements are $L'_{ij} = k_i^{\alpha}k_j^{\alpha}$. From the identity

$$\det(DL' - \lambda I) = \det(D^{\frac{1}{2}}L'D^{\frac{1}{2}} - \lambda I) \tag{5}$$

valid for any λ, where "det" denotes the determinant and I is the $N\times N$ identity matrix, we have that the spectrum of eigenvalues of matrix G is equal to the spectrum of a symmetric matrix defined as

$$H = D^{\frac{1}{2}}L'D^{\frac{1}{2}}. \tag{6}$$

As a result, the eigenvalues of matrix G are all nonnegative real and the smallest eigenvalue is always zero.

III. SIMULATIONS

In our coupling method, giving the parameter β, for $\alpha > 0$, nodes with large degrees have stronger influence, and for $\alpha < 0$, nodes that bear few edges are more influential. Parameter β is exploited to eliminate the discrepancies between the coupling signals that each node receive: Given α, when $\beta = 1$, each node receives the equal quantum of signals, when $\beta < 1$, nodes that have larger sum of neighbors' degrees are influenced more strongly, and when $\beta > 1$, the contrary situation occurs.

Figure 1(a) shows the numerical values of eigenratio R on the parameter space (α, β) for the well-known Barabási-Albert (BA) networks [6]. To clearly exhibit the effects of α and β on R, we report the values of R as a function of α for different β in figure 1(b). No matter what value the parameter β takes, there exists a region of α, in which the eigenratio R is smaller than that of

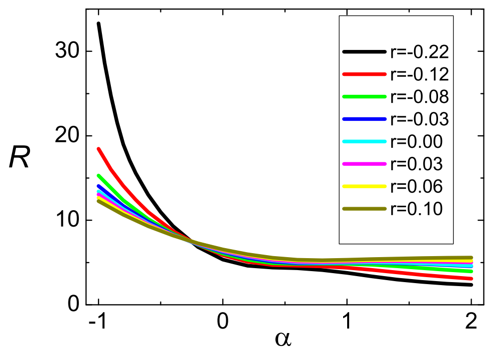

FIG. 4: (color online) The eigenratio R vs the parameter α at $\beta = 1.0$ for the generalized BA networks with different assortative coefficients r. In all cases, the average degree is $\bar{k} = 6$, and the network size is $N = 1024$. The data are obtained over 10 realizations of network configurations.

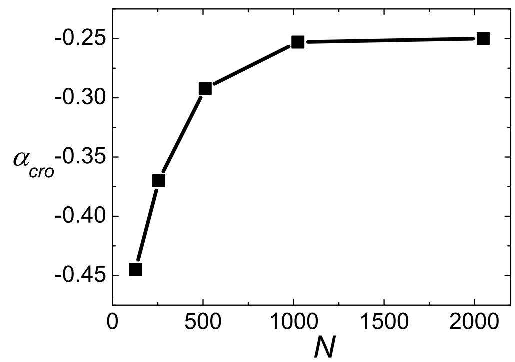

FIG. 6: The crossed value of α vs the network size N at $\beta = 1.0$ for the generalized BA networks. In all cases, the average degree is $\bar{k} = 6$.

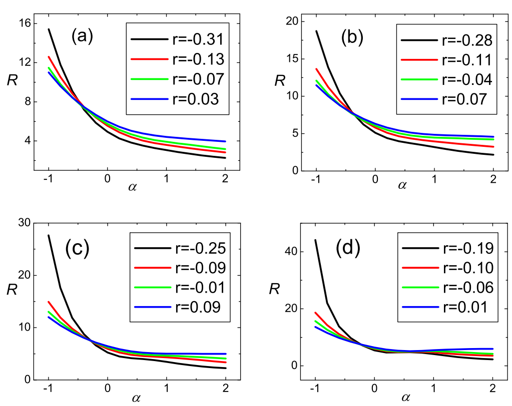

FIG. 5: (color online) Ratio R vs the parameter α at $\beta = 1.0$ for the generalized BA networks with different assortative coefficients r. In each plot, the average degree is $\bar{k} = 6$, and the network size is $N = 128$ (a), $N = 256$ (b), $N = 512$ (b), $N = 2048$ (d). The data are obtained over 10 realizations of network configurations.

the case $\alpha = 0$. That is to say, when proper parameters are chosen, our coupling method can be even better than the MZK method. Similar to the results obtained from MZK method, $\beta = 1.0$ corresponds to the optimal case (i.e. the highest synchronizability). Hereinafter, we concentrate on the case of $\beta = 1$.

Note that, in the limit $\alpha = +\infty$ $(-\infty)$, each node is only influenced by the neighbor having the largest (smallest) degree. The similar situation as mentioned in Ref. [19] appears: The original network approaches to a new configuration that is connected by some effective directed edges [28], and the new network, named the *equivalent network*, may be either connected or disconnected. In the disconnected case, the eigenratio R will approach to infinite, while in the connected case, the eigenratio equals to 2 or some other larger constants. Figure 2 illuminates three simple equivalent networks with $\alpha = +\infty$; the former two are connected, and the third one is disconnected. Their eigenratio are 2, 6.8284 and $+\infty$, respectively. Figure 3 shows the changes of eigenratio R with the parameter α with $\beta = 1$ of different network configurations. When $\alpha > 0.4$, the eigenratios for different configurations go apart: Some approach 2 or other constants not much larger than 2, while some go to infinity, due to whether the equivalent networks are connected or not. In addition, from the simulation, we find that with the increasing of network size, the proportion of networks being disconnected when $\alpha = \infty$ $(-\infty)$ will increase sharply.

Next, we investigate the effects of degree-degree correlation on the network synchronizability [29]. The correlated networks are generated by an extended BA algorithm [30, 31]: Starting from m_0 fully connected nodes, then, at each time step, a new node is added to the network and m $(< m_0)$ previously existing nodes are chosen to be connected to it with probability

$$p_i \propto \frac{k_i + k_0}{\sum_j (k_j + k_0)} \tag{7}$$

where p_i and k_i denote the choosing probability and degree of node i, respectively. By varying the free parameter k_0 $(> -m)$, one can obtain the scale-free networks with different assortative coefficients r [32, 33].

Fig. 4 shows the relationship between eigenratio R and the parameter α for different assortative coefficients given $\beta = 1$. Interestingly, there exists a unique cross point at $\alpha_{cro} \approx -0.25$. When $\alpha < \alpha_{cro}$, the stronger assortative of network predicts better synchronizability, while when $\alpha > \alpha_{cro}$, contrary phenomenon appears. In Fig. 5, we report the simulation results for networks with different sizes, which heightens the reliability of the existence of

this crossed behavior. Fig. 6 exhibits the cross point α_{cro} as a function of the network size N: It increases when N is small, and will get steadily at about 0.25 for sufficiently large N. Although it is interesting, unfortunately, we are not able to provide a theoretical explanation about this phenomenon.

IV. CONCLUSION AND DISCUSSION

The hub nodes of a highly heterogeneous network always play the major roles in determining the dynamical behaviors of the network. In synchronizing process, the hub nodes simultaneously have two effects. On the one hand, the throughput of these hub nodes are too heavy, thus they will hinder the coupling signals' transmission. On the other hand, they have great controlling capability for their large number of coupling neighbors. The MZK coupling method has taken into account the former point, and can predict much better synchronizability than the uniform coupling method. The present method further considers both aspects, performs even better than MZK method, and shows that synchronizability is maximum for a set of parameters that is equivalent to the method introduced in Ref. [16].

Some previous works [34, 35, 36, 37] suggested that there exists some essentially common features between network traffic and synchronization on a dynamical level since the performance of them both are mainly determined by the maximal betweenness, and many methods used to enhance the network synchronizability can also improve the traffic conditions. Therefore, a natural question raises: will the network throughput increase if each node tends to receive information packets from the large-degree neighbors?

Acknowledgments

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